Def (Circulant Matrix)

$A \in \mathbb{R}^{N \times N}$ is said to be circulant

if
$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_N \\ a_N & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_N & a_1 & \cdots & a_{N-1} \\ \vdots & & & \ddots & \vdots \\ a_2 & & \cdots & & a_1 \end{bmatrix}$$
(The rows are shifting right)

or equivalently,
$$A = \begin{bmatrix} a'_1 & a'_N & a'_{N-1} & \cdots & a'_2 \\ a'_2 & a'_1 & a'_N & & a'_3 \\ \vdots & a'_2 & a'_1 & & \vdots \\ & \vdots & & \ddots & \\ a'_N & a'_N & \cdots & & a'_1 \end{bmatrix}$$
(The cols are shifting down)

Def. (Block Circulant Matrix)

$H \in \mathbb{R}^{MN \times MN}$ is Block Circulant

if
$$H = \begin{bmatrix} H_1 & H_2 & H_3 & \cdots & & H_M \\ H_M & H_1 & H_2 & \cdots & & H_{M-1} \\ & H_M & H_1 & H_2 & \cdots & H_{M-2} \\ \vdots & & & & \ddots & \vdots \\ H_2 & & & \cdots & & H_1 \end{bmatrix}$$

where each $H_i \in \mathbb{R}^{N \times N}$ are circulant matrices.

A Block Circulant matrix can be understand as "A Circulant matrix of circulant matrices".

$k, f \in \mathbb{R}^{N \times N}$

$O(f) := k * f = g$

Show that Transform matrix H for O is block circular.

$$\begin{bmatrix} \vec{g}_1 \\ \vdots \\ \vec{g}_N \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1N} \\ H_{21} & H_{22} & & \vdots \\ \vdots & & \ddots & \\ H_{N1} & \cdots & & H_{NN} \end{bmatrix} \begin{bmatrix} \vec{f}_1 \\ \vdots \\ \vec{f}_N \end{bmatrix}$$

where each $H_{ij} \in \mathbb{R}^{N \times N}$

$\vec{g}_1 = H_{11} \vec{f}_1 + H_{12} \vec{f}_2 + \cdots + H_{1N} \vec{f}_N$

$\vec{g}_i = H_{i1} \vec{f}_1 + H_{i2} \vec{f}_2 + \cdots + H_{iN} \vec{f}_N$

Recall the geometric meaning of $k * f$:

periodically extend f,

rotate k by $180°$ to get k'

and overlay k' on f.

$$k' = \begin{bmatrix} | & | & & | \\ \vec{k}'_1 & \vec{k}'_2 & \cdots & \vec{k}'_N \\ | & | & & | \end{bmatrix}$$

k'

$S_{11} \leftarrow$

k'

$S_{21} \swarrow$

$$\begin{matrix}
f_{11} & f_{12} & \cdots & f_{1N} & f_{11} & f_{12} & \cdots & f_{1N} \\
f_{21} & f_{22} & \cdots & f_{2N} & f_{21} & f_{22} & \cdots & f_{2N} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
f_{N1} & f_{N2} & \cdots & f_{NN} & f_{N1} & f_{N2} & \cdots & f_{NN} \\
f_{11} & f_{12} & \cdots & f_{1N} & f_{11} & f_{12} & \cdots & f_{1N} \\
f_{21} & f_{22} & \cdots & f_{2N} & f_{21} & f_{22} & \cdots & f_{2N} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
f_{N1} & f_{N2} & \cdots & f_{NN} & f_{N1} & f_{N2} & \cdots & f_{NN}
\end{matrix}$$

$$S_{11} = \vec{k}_1'^{T} \vec{f}_1 + \vec{k}_2'^{T} \vec{f}_2 + \cdots + \vec{k}_N'^{T} \vec{f}_N$$

$$S_{21} = \underbrace{(\vec{k}_1' \downarrow_1)}_{\text{shift down by 1}}^{T} \vec{f}_1 + (\vec{k}_2' \downarrow_1)^{T} \vec{f}_2 + \cdots + (\vec{k}_N' \downarrow_1)^{T} \vec{f}_N$$

shift down by 1 : $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$, $\vec{a} \downarrow_1 = \begin{bmatrix} a_N \\ a_1 \\ a_2 \\ \vdots \\ a_{N-1} \end{bmatrix}$

(Your Tutor's own notation.
Please define it clearly if you want to use it elsewhere.)

$$S_{i1} = (\vec{k}_1' \downarrow_{i-1})^{T} \vec{f}_1 + (\vec{k}_2' \downarrow_{i-1})^{T} \vec{f}_2 + \cdots + (\vec{k}_N' \downarrow_{i-1})^{T} \vec{f}_N$$

$$\vec{s}_1 = \begin{bmatrix} s_{11} \\ s_{21} \\ \vdots \\ s_{N1} \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{k}_1'^{T} \\ (\vec{k}_1' \downarrow_1)^T \\ \vdots \\ (\vec{k}_1' \downarrow_{N-1})^T \end{bmatrix}}_{= H_{11}} \vec{f}_1 + \underbrace{\begin{bmatrix} \vec{k}_2'^{T} \\ (\vec{k}_2' \downarrow_1)^T \\ \vdots \\ (\vec{k}_2' \downarrow_{N-1})^T \end{bmatrix}}_{H_{12}} \vec{f}_2 + \dots + \underbrace{\begin{bmatrix} \vec{k}_N'^{T} \\ (\vec{k}_N' \downarrow_1)^T \\ \vdots \\ (\vec{k}_N' \downarrow_{N-1})^T \end{bmatrix}}_{H_{1N}} \vec{f}_N$$

$$= \begin{bmatrix} k'_{11} & k'_{21} & \dots & k'_{N1} \\ k'_{N1} & k_{11} & k'_{21} & \dots & k'_{N-1,1} \\ k'_{N-1,1} & k'_{N1} & k'_{11} & & \\ & & & \ddots & \end{bmatrix}$$

circular!

k'

$$
\begin{array}{cccccccc}
f_{11} & f_{12} & \cdots & f_{1N} & f_{11} & f_{12} & \cdots & f_{1N} \\
f_{21} & f_{22} & \cdots & f_{2N} & f_{21} & f_{22} & \cdots & f_{2N} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
f_{N1} & f_{N2} & \cdots & f_{NN} & f_{N1} & f_{N2} & \cdots & f_{NN} \\
f_{11} & f_{12} & \cdots & f_{1N} & f_{11} & f_{12} & \cdots & f_{1N} \\
f_{21} & f_{22} & \cdots & f_{2N} & f_{21} & f_{22} & \cdots & f_{2N} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
f_{N1} & f_{N2} & \cdots & f_{NN} & f_{N1} & f_{N2} & \cdots & f_{NN}
\end{array}
$$

$g_{12} \leftarrow$

$g_{22} \leftarrow$

k'

$$g_{12} = \vec{k}'^{T}_{1} \vec{f}_{2} + \vec{k}'^{T}_{2} \vec{f}_{3} + \cdots + \vec{k}'^{T}_{N-1} \vec{f}_{N} + \vec{k}'^{T}_{N} \vec{f}_{1}$$

$$= \vec{k}'^{T}_{N} \vec{f}_{1} + \vec{k}'^{T}_{1} \vec{f}_{2} + \cdots + \vec{k}'^{T}_{N-1} \vec{f}_{N}$$

$$g_{22} = (\vec{k}'_N \downarrow_1)^T \vec{f}_1 + (\vec{k}'_N \downarrow_1)^T \vec{f}_2 + \cdots + (\vec{k}'_{N-1} \downarrow_1)^T \vec{f}_N$$

$$g_{i2} = (\vec{k}'_N \downarrow_{i-1})^T \vec{f}_1 + (\vec{k}'_N \downarrow_{i-1})^T \vec{f}_2 + \cdots + (\vec{k}'_{N-1} \downarrow_{i-1})^T \vec{f}_N$$

$$\vec{g}_2 = \underbrace{\begin{bmatrix} - & \vec{k}'^{T}_{N} & - \\ - & (\vec{k}'_N \downarrow_1)^T & - \\ & \vdots & \\ - & (\vec{k}'_N \downarrow_{N-1})^T & - \end{bmatrix}}_{= H_{21} = H_{1N}} \vec{f}_1 + \underbrace{\begin{bmatrix} - & \vec{k}'^{T}_{1} & - \\ - & (\vec{k}'_1 \downarrow_1)^T & - \\ & \vdots & \\ - & (\vec{k}'_1 \downarrow_{N-1})^T & - \end{bmatrix}}_{= H_{22} = H_{11}} \vec{f}_2 + \cdots + \underbrace{\begin{bmatrix} - & \vec{k}'^{T}_{N-1} & - \\ - & (\vec{k}'_{N-1} \downarrow_1)^T & - \\ & \vdots & \\ - & (\vec{k}'_{N-1} \downarrow_{N-1})^T & - \end{bmatrix}}_{= H_{2N} = H_{1,N-1}} \vec{f}_N$$

Similar approach for all $\vec{g}_i$,

$$\begin{bmatrix} \vec{g}_1 \\ \vec{g}_2 \\ \vdots \\ \vec{g}_N \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1N} \\ H_{21} & H_{22} & \cdots & H_{2N} \\ \vdots & & & \vdots \\ H_{N1} & H_{N2} & \cdots & H_{NN} \end{bmatrix} \begin{bmatrix} \vec{f}_1 \\ \vec{f}_2 \\ \vdots \\ \vec{f}_N \end{bmatrix}$$

$$= \begin{bmatrix} H_{11} & H_{12} & & \cdots & H_{1N} \\ H_{1N} & H_{11} & H_{12} & \cdots & H_{1,N-1} \\ \vdots & & \ddots & & \ddots \end{bmatrix} \begin{bmatrix} \vec{f}_1 \\ \vec{f}_2 \\ \vdots \\ \vec{f}_N \end{bmatrix}$$

$\therefore$ Block - Circulant.

$f \in \mathbb{C}^{N \times N}$

Find a formula for $\hat{\hat{f}}(\alpha, \beta)$.

$$\hat{f}(m,n) = \frac{1}{N^2} \sum_{k,l=0}^{N-1} \sum f(k,l)\, e^{-j\frac{2\pi}{N}(mk+nl)}$$

$$\hat{\hat{f}}(\alpha,\beta) = \frac{1}{N^2} \sum_{m,n}^{N-1} \sum \hat{f}(m,n)\, e^{-j\frac{2\pi}{N}(\alpha m+\beta n)}$$

$$= \frac{1}{N^4} \sum_m \sum_n \sum_k \sum_l f(k,l)\, e^{-j\frac{2\pi}{N}[m(\alpha+k)+n(\beta+l)]}$$

$$= \frac{1}{N^4} \sum_k \sum_l \left[f(k,l) \left(\sum_m \sum_n e^{-j\frac{2\pi}{N}(\alpha+k)m} \cdot e^{-j\frac{2\pi}{N}(\beta+l)n} \right) \right]$$

$$= \frac{1}{N^4} \sum_k \sum_l \left\{ f(k,l) \left(\sum_m e^{-j\frac{2\pi}{N}(\alpha+k)m} \right) \left(\sum_n e^{-j\frac{2\pi}{N}(\beta+l)n} \right) \right\}$$

$$\sum_m e^{-j\frac{2\pi}{N}(\alpha+k)m} = \begin{cases} \sum_m 1, & \text{if } k=-\alpha \\ \dfrac{e^{-j\frac{2\pi}{N}(\alpha+k)\cdot N}-1}{e^{-j\frac{2\pi}{N}(\alpha+k)}-1}, & \text{if } k\neq-\alpha \end{cases}$$

$$= \begin{cases} N, & \text{if } k=-\alpha \\ 0, & \text{if } k\neq-\alpha \end{cases}$$

Similarly,

$$\sum_n e^{-j\frac{2\pi}{N}(\beta+l)n} = \begin{cases} N, & \text{if } l=-\beta \\ 0, & \text{if } l\neq-\beta. \end{cases}$$

$$\therefore \quad \hat{\hat{f}}(\alpha, \beta) = \frac{1}{N^2} f(-\alpha, -\beta)$$

i.e.

$$\widehat{\widehat{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}} = \frac{1}{N^2} \cdot \begin{bmatrix} \text{p} & \text{ɔ} \\ \text{q} & \text{ɐ} \end{bmatrix}$$

$A \in \mathbb{R}^{M \times N}$, rank$(A) = r$

$A = U \Sigma V^T$

A_k rank k approx. by SVD

Let $B \in \mathbb{R}^{M \times N}$, rank$(B) = k$.

$\vdash_1$: $\| A - A_k \|_2 \leq \| A - B \|_2$

$\vdash_2$: $\| A - A_k \|_F \leq \| A - B \|_F$

where for $C \in \mathbb{R}^{M \times N}$

$$\| C \|_2 := \sup_{\vec{x} \in \mathbb{R}^N} \frac{\| \underbrace{C \vec{x}}_{\in \mathbb{R}^M} \|_2}{\| \underbrace{\vec{x}}_{\in \mathbb{R}^N} \|_2}$$

$$= \sup_{\substack{\vec{x} \in \mathbb{R}^N \\ \| \vec{x} \|_2 = 1}} \| C \vec{x} \|_2$$

$$= \sigma_1 (C).$$

pf of $\vdash_1$

$\text{rank}(B) = k \le r = \text{rank}(A)$

Then B can be written as XY^T

where $X \in \mathbb{R}^{M\times k}$, $Y \in \mathbb{R}^{N\times k}$

(You can consider $B = PSQ^T$ (SVD)

and take X to be the first k cols of PS,

take Y to be the first k cols of Q)

$A = U\Sigma V^T$ (SVD)

$\vec{v}_1, \cdots, \vec{v}_{k+1} \in \mathbb{R}^N$ first $k+1$ cols of V.

Since $\text{rank}(Y^T) = k$,

and $\dim(\text{span}(\vec{v}_1, \cdots, \vec{v}_{k+1})) = k+1$,

There must be a non-trivial linear combination

of $\vec{v}_1, \cdots, \vec{v}_{k+1}$,

written as $\vec{w} = \gamma_1 \vec{v}_1 + \cdots + \gamma_{k+1}\vec{v}_{k+1}$,

falls into the null space of Y^T.

$\vec{w} \neq 0$, $\tilde{w} := \frac{\vec{w}}{\|\vec{w}\|}$

$$\| A - B \|_2^2$$

$$= \left(\sup_{\substack{\vec{z} \in \mathbb{R}^N \\ \|\vec{z}\|=1}} \| (A - B) \vec{z} \|_2 \right)^2$$

$$\geq \| (A - B) \tilde{w} \|_2^2$$

$$= \| U \Sigma V^T \tilde{w} - X Y^T \tilde{w} \|_2^2$$

$$= \| \Sigma V^T (\gamma_1 \vec{v}_1 + \cdots + \gamma_{k+1} \vec{v}_{k+1}) \|_2^2$$

$$= \| \Sigma (\gamma_1 \vec{e}_1 + \cdots + \gamma_{k+1} \vec{e}_{k+1}) \|_2^2$$

$$= \gamma_1^2 \sigma_1^2 + \cdots + \gamma_{k+1}^2 \sigma_{k+1}^2$$

$$\geq \sigma_{k+1}^2 (\gamma_1^2 + \cdots + \gamma_{k+1}^2)$$

$$= \sigma_{k+1}^2$$

$$= \sigma_1 (A - A_k)^2$$

$$= \| A - A_k \|_2^2$$

$\vdash_1$ proved.

pf of $\vdash_2$

Recall $\operatorname{rank}(A) = r$,

$$\| A - A_k \|_F^2 = \sum_{i=k+1}^{r} \sigma_i^2$$

Suppose $A = A' + A''$

Note $\sigma_1(M_1 + M_2) = \| M_1 + M_2 \|_2$

(triangle inequality of norm) $\leq \| M_1 \|_2 + \| M_2 \|_2$

$= \sigma_1(M_1) + \sigma_1(M_2)$.

$\sigma_i(A') + \sigma_j(A'')$

$= \sigma_1(A' - A'_{i-1}) + \sigma_1(A'' - A''_{j-1})$

rank $i-1$ approx. by SVD

rank $j-1$ approx. by SVD

(triangle inequality) $\geq \sigma_1(A' - A'_{i-1} + A'' - A''_{j-1})$

$= \sigma_1(A - (A'_{i-1} + A''_{j-1}))$

rank at most $i+j-2$

by $\vdash_1$, $\geq \sigma_1(A - A_{i+j-2})$ ← best rank $i+j-2$ approx. in 2-norm.

$= \sigma_{i+j-1}(A)$

Take $A' = A - B$, $A'' = B$,

for $i \geq 1$, $j = k+1$,

$\sigma_i(A-B) + \underline{\underline{\sigma_j(B)}} \geq \sigma_{i+k}(A)$

$= \sigma_{k+1}(B) = 0$ as $\operatorname{rank}(B) = k$

Then $\| A - B \|_F^2 = \sum_{i=1}^{\min\{M,N\}} \sigma_i (A - B)^2$

$$\geq \sum_{i=1}^{r-k} \sigma_{i+k}(A)^2$$

$$= \sum_{i=k+1}^{r} \sigma_i (A)^2$$

$$= \| A - A_k \|_F^2$$

$\therefore$ F_2 proved.